ENGG2012B Advanced Engineering Mathematics

Notes on PageRank Algorithm

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The Pagerank algorithm was invented by Page and Brin around 1998 and used in the prototype of Google's search engine. The objective is to estimate the popularity, or the importance, of a webpage, based on the interconnection of the web. The rationale behind it is (i) a page with more incoming links is more important than a page with less incoming links, (ii) a page with a link from a page which is known to be of high importance is also important.

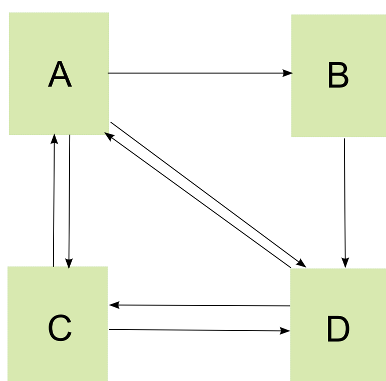
In this note, we study the convergence of the Pagerank algorithm from matrix's point of view. In practice the number of web pages is huge. In this notes, only examples of small size will be given.

1 Simplified Pagerank algorithm

The connections between pages is represented by a graph. A node represents a webpage and an arrow from page A to page B means that there is a link from page A to page B. The number of out-going links is an important parameter. We use the notation "out-degree of a node" to stand for the number of out-going links contained in a page. This graph is usually referred to as the *web graph*. Each node in the graph is identified with a page. We will use the term "node" and "page" interchangeably.

We following the notations in the wikipedia page for the Pagerank algorithm and let $L(p)$ be the number of out-going links in a page p.

Example 1: There are four pages. Page A contains a link to page B, a link to page C, and a link to page D. Page B contains one single link to page D. Page C points to pages A and D, and page D points to pages A and C. They are represented by the following graph. We have $L(\mathsf{A}) = 3$, $L(\mathsf{B}) = 1$ and $L(\mathsf{C}) = L(\mathsf{D}) = 2$.



Let N be the total number of pages. We create an $N \times N$ matrix $\mathbf{A}$ by defining the (i, j)-entry as

$$a_{ij} = \begin{cases} \frac{1}{L(j)} & \text{if there is a link from } j \text{ to } i, \\ 0 & \text{otherwise.} \end{cases}$$

In Example 1, the matrix $\mathbf{A}$ is the 4×4 matrix

$$\begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 1/2 & 0 \end{bmatrix}.$$

Note that the sum of the entries in each column is equal to 1.

In general, a matrix is said to be *column-stochastic* if the entries are non-negative and the sum of the entries in each column is equal to 1. The matrix **A** is by design a column-stochastic matrix, provided that each page contains at least one out-going link.

The simplified Pagerank algorithm is:

> Initialize $\mathbf{x}$ to an $N \times 1$ column vector with non-negative components, and then repeatedly replace $\mathbf{x}$ by the product $\mathbf{Ax}$ until it converges.

We call the vector $\mathbf{x}$ the *pagerank vector*. Usually, we initialize it to a column vector whose components are equal to each other.

We can imagine a bored surfer who clicks the links in a random manner. If there are k links in the page, he/she simply picks one of the links randomly and goes the selected page. After a sufficiently long time, the N components of the pagerank vector are directly proportional to the number of times this surfer visits the N web pages.

For example, in Example 1, we let the components of the vector $\mathbf{x}$ be x_{A}, x_{B}, x_{C} and x_{D}. Initialize $\mathbf{x}$ to be the all-one column vector, i.e.,

$$\mathbf{x} = \begin{bmatrix} x_{\mathsf{A}} \\ x_{\mathsf{B}} \\ x_{\mathsf{C}} \\ x_{\mathsf{D}} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The evolution of the pagerank vector is shown in the following table.

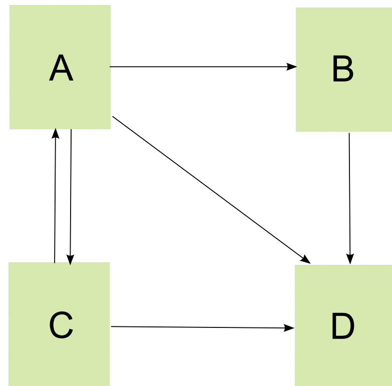
iteration	x_{A}	x_{B}	x_{C}	x_{D}
0	1	1	1	1
1	1	0.3333	0.8333	1.8333
2	1.3333	0.3333	1.25	1.0833
3	1.667	.4444	.9861	1.4028
4	1.1944	0.3889	1.0903	1.3264
5	1.2083	0.3981	1.0613	1.3322
6	1.1968	0.4028	1.0689	1.3316
7	1.2002	0.3989	1.0647	1.3361

We observe that the algorithm converges quickly in this example. Within 10 iterations, we can see that page D has the highest rank. In fact, page D has 3 incoming links, while the others have either 1 or 2 incoming links. It conforms with the rationale of the pagerank algorithm that a page with larger incoming links has higher importance.

2 How to handle dangling node

A node is called a *dangling node* if it does not contain any out-going link, i.e., if the out-degree is zero. For instance, node D in the next example is a dangling node.

Example 2:



The associated matrix $\mathbf{A}$ is

$$\begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 1/2 & 0 \end{bmatrix}.$$

We note that the entries in the last column are all zero, hence this matrix is not column-stochastic.

The simplified Pagerank algorithm collapses if there is dangling node in the web graph. As an example, if we initialize the vector $\mathbf{x}$ to the all-one vector, the simplified Pagerank algorithm gives

iteration	x_{A}	x_{B}	x_{C}	x_{D}
0	1	1	1	1
1	0.5	0.3333	0.8333	1.8333
2	0.1667	0.1667	0.1667	0.6667
3	0.0833	0.0556	0.0556	0.3056
4	0.0278	0.0278	0.0278	0.1111
5	0.0139	0.0093	0.0098	0.0509

The pagerank vector will converge to zero ultimately.

One remedy is to modify the simplified Pagerank algorithm by replacing any all-zero column by

$$\begin{bmatrix} 1/N \\ 1/N \\ \vdots \\ 1/N \end{bmatrix}.$$

This adjustment is justified by modeling the behaviour of a web surfer, who after reading a page with no out-going link, he/she will jump to a random page. He/she simply picks one of the N pages randomly with equal probability. This model may not be realistic, but it simplifies the computation of the algorithm.

In matrix format, we create a matrix $\bar{\mathbf{A}}$, whose column is the same as $\mathbf{A}$ except the columns corresponding to the dangling pages. Formally, we define the entries of matrix $\bar{\mathbf{A}}$ by

$$\bar{a}_{ij} = \begin{cases} \frac{1}{L(j)} & \text{if there is a link from node } j \text{ to node } i, \\ \frac{1}{N} & \text{if node } j \text{ is a dangling node,} \\ 0 & \text{otherwise,} \end{cases}$$

where N is the total number of pages.

In Example 2, the matrix $\bar{\mathbf{A}}$ is

$$\begin{bmatrix} 0 & 0 & 1/2 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 1 & 1/2 & 1/4 \end{bmatrix}.$$

In the Pagerank algorithm, we use $\bar{\mathbf{A}}$ instead of $\mathbf{A}$, and replace $\mathbf{x}$ by $\bar{\mathbf{A}}\mathbf{x}$ instead. In Example 2, if we use $\bar{\mathbf{A}}$, we get

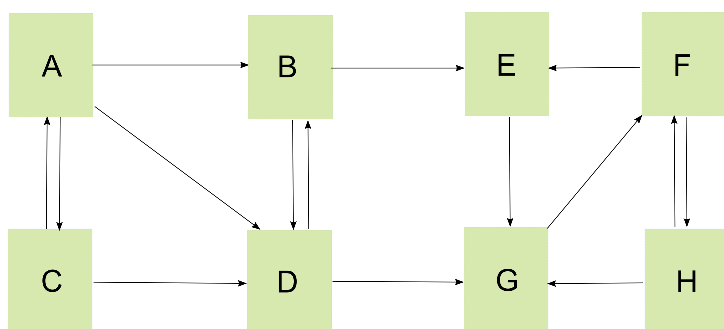
iteration	x_A	x_B	x_C	x_D
0	1	1	1	1
1	0.75	0.5833	0.5833	2.0833
2	0.8125	0.7708	0.7708	1.6458
3	0.7969	0.6823	0.6823	1.8385
4	0.8008	0.7253	0.7253	1.7487
5	0.7998	0.7041	0.7041	1.7920

We see that page D has the highest ranking.

3 How to handle reducible web graph

Consider the following web graph consisting of eight web pages.

Example 3:



There is no dangling node. But, once a surfer arrives at pages E, F, G or H, he/she will get stuck in these four pages. The matrix $\mathbf{A}$ of this web graph is

$$\begin{bmatrix} 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \end{bmatrix}$$

If we initialize the pagerank vector $\mathbf{x}$ to be the 8×1 column vector with components all equal to $1/8$, then the Pagerank algorithm gives

iteration	x_{A}	x_{B}	x_{C}	x_{D}	x_{E}	x_{F}	x_{G}	x_{H}
0	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125
1	0.0625	0.1042	0.0417	0.1667	0.125	0.1875	0.25	0.0625
2	0.0208	0.1042	0.0208	0.0937	0.1458	0.2813	0.2396	0.0938
3	0.0104	0.0538	0.0069	0.0697	0.1927	0.2865	0.2396	0.1406
4	0.0035	0.0382	0.0035	0.0339	0.1701	0.3099	0.2977	0.1432
5	0.0017	0.0181	0.0012	0.0220	0.1740	0.3694	0.2587	0.1549
6	0.0006	0.0116	0.0006	0.0102	0.1937	0.3362	0.2625	0.1847
7	0.0003	0.0053	0.0002	0.0063	0.1739	0.3549	0.2912	0.1681
8	0.0001	0.0032	0.0001	0.0028	0.1801	0.3752	0.2610	0.1774

We can see that the ranking of pages A to D drop to zero eventually. But page D has three incoming links and should have some nonzero importance. The Pagerank algorithm does not work in this example.

This pathological web graph belongs to the category of *reducible graph*. In general, a graph is called *irreducible* if for any pair of distinct nodes, we can start from one of them, follow the links in the web graph and arrive at the other node, and *vice versa*. A graph which is not irreducible is called reducible. In Example 3, there is no path from E to B, and no path from G to D. The graph is therefore reducible.

In order to calculate page ranks properly for a reducible web graph, Page and Brin proposed that take a weighted average of the matrix $\bar{\mathbf{A}}$ with an all-one $N \times N$ matrix. Define the matrix

$$\mathbf{M} = d\bar{\mathbf{A}} + \frac{1-d}{N}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

where $\bar{\mathbf{A}}$ is the modified matrix defined in the previous section, and d is a number between 0 and 1. The constant d is usually called the *damping factor* or the *damping constant*. The default value of d is 0.85 in Google.

With d set to 0.85, the matrix $\mathbf{M}$ for the web graph in Example 3 is

$$\mathbf{M} = \begin{bmatrix} 0.0187 & 0.0187 & 0.4437 & 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.0187 \\ 0.3021 & 0.0187 & 0.0187 & 0.4437 & 0.0187 & 0.0187 & 0.0187 & 0.0187 \\ 0.3021 & 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.0187 \\ 0.3021 & 0.4437 & 0.4437 & 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.0187 \\ 0.0187 & 0.4437 & 0.0187 & 0.0187 & 0.0187 & 0.4437 & 0.0187 & 0.0187 \\ 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.8688 & 0.4437 \\ 0.0187 & 0.0187 & 0.0187 & 0.4437 & 0.8688 & 0.0187 & 0.0187 & 0.4437 \\ 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.0187 & 0.4437 & 0.0187 & 0.0187 \end{bmatrix}$$

Using this matrix in each iteration, the evolution of the pagerank vector is

iteration	x_{A}	x_{B}	x_{C}	x_{D}	x_{E}	x_{F}	x_{G}	x_{H}
0	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125
1	0.0719	0.1073	0.0542	0.1604	0.1250	0.1781	0.2313	0.0719
2	0.0418	0.1073	0.0391	0.1077	0.1401	0.2459	0.2237	0.0945
3	0.0354	0.0764	0.0306	0.0928	0.1688	0.2491	0.2237	0.1232
4	0.0317	0.0682	0.0288	0.0742	0.1571	0.2613	0.2541	0.1246
5	0.0310	0.0593	0.0277	0.0690	0.1588	0.2877	0.2368	0.1298
6	0.0305	0.0568	0.0275	0.0645	0.1662	0.2752	0.2382	0.1410
7	0.0304	0.0548	0.0274	0.0633	0.1598	0.2811	0.2474	0.1357
8	0.0304	0.0543	0.0274	0.0623	0.1615	0.2867	0.2392	0.1382

We see that the page ranks of pages A to D do not vanish.

4 Convergence of the Pagerank algorthm

The matrix $\mathbf{M}$ is column-stochastic by the design of the Pagerank algorithm. Also, all entries are positive. We can invoke the Perron-Frobenius theorem of matrix in the analysis of the convergence of the Pagerank algorithm. The Perron-Frobenius theorem states that for any column-stochastic matrix with positive entries, we can always find 1 as an eigenvalue, and all other eigenvalues have magnitude strictly less than 1. Moreover, 1 is a simple eigenvalue ("simple" means that 1 is a root, but not a repeated root, of the characteristic polynomial.)

The Perron-Frobenius theorem is a fundamental theorem for matrices with positive entries. We shall take it for granted in this notes.

Let λ_i, for $i = 1, 2, \dots, N$ be the eigenvalues of the matrix $\mathbf{M}$, sorted in decreasing order of absolute values. (The eigenvalues may be complex, so we have to take the absolute value.) By Perron-Frobenius theorem, we have

$$1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \geq \cdots \geq |\lambda_N|.$$

At this point, we make a simplifying assumption that all eigenvalues are distinct, i.e., we can find N distinct eigenvalues for the $N \times N$ matrix $\mathbf{M}$. We use a fact from linear algebra that if the eigenvalues of a matrices are distinct, we can find an eigenvector for each of the N eigenvalues, and the N eigenvectors are linearly independent.

Let $\mathbf{v}_i$, for $i = 1, 2, \dots, N$, be the eigenvectors, corresponding to the eigenvalues λ_i, respectively. Let $\mathbf{P}$ be an $N \times N$ matrix with the j-th column equal to $\mathbf{v}_j$, for $j = 1, 2, \dots, N$,

$$\mathbf{P} := \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_N \end{bmatrix},$$

Since the column of $\mathbf{P}$ are linearly independent, by the last theorem in Lecture notes 9, the determinant of $\mathbf{P}$ is non-zero, and hence the inverse of $\mathbf{P}$ exists.

Let $\mathbf{D}$ be a diagonal matrix, whose diagonal entries are the eigenvalues λ_i,

$$\mathbf{D} := \begin{bmatrix} 1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_N \end{bmatrix}$$

We can put the defining equation of eigenvectors, $\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i$, for $i = 1, 2, \dots, N$, together in a single matrix equation

$$\mathbf{MP} = \mathbf{PD},$$

or equivalently

$$\mathbf{M} = \mathbf{PDP}^{-1}.$$

The last line relies on the fact that $\mathbf{P}$ is invertible. The factorization of $\mathbf{M}$ as $\mathbf{PDP}^{-1}$ is called the diagonalization of $\mathbf{M}$.

Let $\mathbf{x}_0$ be the initial pagerank vector. The pagerank vector after k iterations is given by

$$\begin{aligned} \mathbf{M}^k \mathbf{x}_0 &= (\mathbf{PDP}^{-1})^k \mathbf{x}_0 \\ &= \underbrace{(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) \cdots (\mathbf{PDP}^{-1})}_{k \text{ times}} \mathbf{x}_0 \\ &= \mathbf{PD}^k \mathbf{P}^{-1} \mathbf{x}_0. \end{aligned}$$

When $k \to \infty$, all diagonal entries in $\mathbf{D}^k$, except the first one, converge to zero.

$$\lim_{k\to\infty} \mathbf{D}^k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

After running the Pagerank algorithm for a long time, the pagerank vector will converge to

$$\mathbf{P} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{P}^{-1}\mathbf{x}_0 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{P}^{-1}\mathbf{x}_0,$$

which is a scalar multiple of the eigenvector $\mathbf{v}_1$. Hence, we can conclude that the Pagerank with damping factor, under the assumption that all eigenvalues of $\mathbf{M}$ are distinct, converges to an eigenvector corresponding to the largest eigenvalue $\lambda_1 = 1$.

We can find the eigenvalues of $\mathbf{M}$ in Matlab by the command `eig`. After typing the matrix $\mathbf{M}$ in Matlab, we can type

```
eig(M)
```

to get the eigenvalues of $\mathbf{M}$. For the matrix $\mathbf{M}$ in Example 3 with damping constant 0.85, the eigenvalues are

```
    1.0000
-0.4250 + 0.6010i
-0.4250 - 0.6010i
0.4250
0.3470
-0.4250
-0.3470
0.0000
```

As expected, the largest eigenvalue is 1, and the rests have absolute value strictly less than 1.

If we want the matrix $\mathbf{P}$, we can also get it by the Matlab command

```
[P D] = eig(M)
```

This will return two matrices. The matrix `D` is a diagonal matrix whose diagonal entries are the eigenvalues. The columns in the matrix `P` are the corresponding eigenvectors, normalized so that the norm of each column is equal to 1. The first column of `P` is the eigenvector corresponding to eigenvalue 1. We can extract the first column by `P(:,1)`, and normalize it by division by `sum(P(:,1))`

```
>> P(:,1)/sum(P(:,1))

ans =
0.0304
0.0536
0.0274
0.0618
0.1621
0.2836
0.2419
0.1393
```

We can compare it with the last row in the last table in the previous section.

What happens when the eigenvalues are not distinct? More complicated analysis involving the Jordan canonical form will be needed in this case, but we will not go into the details.

A successful algorithm or method usually has a rigorous mathematical theory behind it. In probability theory, the Pagerank algorithm is essentially a Markov chain. The Perron-Frobenius theorem is a standard tool in the analysis of convergence in Markov chain.