Abstract Explicit constructions of optimal conflict-avoiding codes of weight three and even length are recently obtained. For odd length, the construction of optimal conflict-avoiding codes of weight three in general remains open. Previous results by Levenshtein consider the special case of conflict-avoiding codes consisting of equi-difference codewords only. In this paper, we take the non-equidifference codewords into account, and reduce the problem to hypergraph matching problem. Some new optimal conflict-avoiding codes are presented.

key words: Conflict-avoiding codes, protocol sequences, collision channel without feedback, hypergraph matching.

1. Introduction

Let \( \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \) denote the ring of residues mod \( n \). Given a non-empty subset \( S \) of \( \mathbb{Z}_n \), we define the set of differences by

\[
\Delta(S) := \{a - b \in \mathbb{Z}_n : a, b \in S\}.
\]

The subtraction in the above definition is performed modulo-\( n \), and 0 is always an element in \( \Delta(S) \). We remark that \( \Delta(S) \) is a set, so that order and multiplicity of elements are irrelevant. Let \( \mathcal{P}_w(\mathbb{Z}_n) \) be the set of all subsets of \( \mathbb{Z}_n \) with cardinality \( w \). A collection \( \mathcal{C} \) of subsets in \( \mathcal{P}_w(\mathbb{Z}_n) \) is called a conflict-avoiding codes (CAC) of weight \( w \) and length \( n \) if for any two subsets \( A \) and \( B \) in \( \mathcal{C} \), we have \( \Delta(A) \cap \Delta(B) = \{0\} \). In other words, if a nonzero element \( d \) in \( \mathbb{Z}_n \) appears as a difference between two elements in a subset in \( \mathcal{C} \), then no other subset in \( \mathcal{C} \) contains a pair of elements whose difference is equal to \( d \). A subset in a CAC is also called a codeword, and the cardinality is also called the weight. Since adding a constant to every element in a codeword \( A \) does not affect \( \Delta(A) \), we assume without loss of generality that 0 \( \in A \) for all \( A \in \mathcal{C} \). We will use the notation \(|S|\) to denote the cardinality of a set \( S \).

Conflict-avoiding codes can be applied in multiple-access system for the collision channel without feedback. In this application, it is assumed that time is divided into time slots, and each user transmits data packets within the duration of the time slots. The channel is modeled as a collision channel, meaning that if two or more users transmit in the same slot, then a collision is incurred and no information message can be decoded. However if only one user transmits in a time slot, the received packet is assumed error-free. Suppose that there are a number of users who want to send data to a common destination, but there is no feedback path from the destination to the users.

We construct binary sequences by treating the subsets in a conflict-avoiding code as the characteristic sets, the resulting binary sequences are called protocol sequences. Each user is statically assigned a protocol sequence, so that the protocol sequences of two different users are cyclically distinct. Each user switches between two states: active and inactive. When a user become active, it reads out the protocol sequence periodically, once per slot duration, and sends a packet when the sequence value is equal to 1, but remains silent when the sequence value is equal to 0. The length \( n \) of a CAC is thus also called the sequence period. Each active users is in the active state for at least one sequence period. When a user has finished sending packets and has nothing more to send for the time being, he switches to the inactive state. Each user stays inactive for at least one sequence period. As no collaboration among the users is required in this protocol, the users may not become active at the same time. The lack of centralized control and user cooperation incur non-zero relative delay offsets of the protocol sequences. If protocol sequences constructed from CAC are used, then, there are at most one collision between each pair of active users in a sequence period. It is guaranteed that whenever the number of simultaneously active users is less than or equal to the number of ones in a period, each active user can send at least one packet to the destination without collision, regardless of the relative delay offsets.

This kind of multiple-access protocol can be applied to alarm system for example. There is a number of nodes distributed in a geographical area. A node sends an alarming message to the data collecting node using the protocol described in the previous paragraph. The period \( n \) measures the maximum delay one has to wait until the message can reach the data sink successfully, and the number of codewords in a CAC equals the total number of nodes that the system can support. The parameter \( w \) is the maximum number of simultaneously active users. The objective is to maximize the number of potential users, given the sequence period \( n \) and the number of active users \( w \).

A conflict-avoiding codes of length \( n \) and weight \( w \) is denoted by \( \text{CAC}(n, w) \). A \( \text{CAC}(n, w) \) is said to be optimal if it has the largest number of codewords among all CACs of length \( n \) and weight \( w \). The number of codewords in an optimal \( \text{CAC}(n, w) \) is denoted by \( M(n, w) \). We present an example for length \( n = 13 \) and weight \( w = 3 \). The three codewords \{0, 1, 2\}, \{0, 3, 6\} and \{0, 4, 8\} form a conflict-avoiding codes of length 13 and weight 3. The corresponding protocol sequences are 1110000000000, 1001001000000, 1000100100000. It can be verified that

\[
\Delta(\{0, 1, 2\}) = \{0, 1, 2, 12, 13\} \\
\Delta(\{0, 3, 6\}) = \{0, 3, 6, 7, 10\} \\
\Delta(\{0, 4, 8\}) = \{0, 4, 5, 8, 9\}.
\]

A codeword of weight 3 is said to be equi-difference or centered if it can be written as \( \{0, \pm a\} \) for some nonzero element \( a \) in \( \mathbb{Z}_n \), or equivalently, if it can be written as

\[
\{0, a, b, a + b, a - b, a + a - b, a - a + b\}.
\]
\[ \{0, a, 2a\} \] after some translation. The element \( a \) is called the \textit{generator} of the centered codeword \( \{0, \pm a\} \). We note that the codewords in the previous example are all equi-difference. If a conflict-avoiding code \( \mathcal{C} \) consists of equi-difference codewords only, we say that \( \mathcal{C} \) is equi-difference. Among the collection of equi-difference CAC, the largest possible number of codewords is denoted by \( M^e(n) \). We will use the adjectives “equi-difference” and “centered” interchangeably.

In [5, 6], Levenshtein and Tonchev prove that \( M(n, 3) \leq (n + 1)/4 \), and
\[
M^e(n, 3) = M(n, 3) = (n - 2)/4 \quad \text{if } n \equiv 2 \text{ mod } 4. \quad (1)
\]
For \( n \) which is a multiple of 4, Jimbo \textit{et al}. obtain a better upper bound on the number of codewords [4]: for \( n = 4t \), we have
\[
M(n, 3) = \begin{cases} 
7n/32 & \text{if } t \equiv 0 \text{ mod } 8 \\
(7n + 4)/32 & \text{if } t \equiv 1 \text{ mod } 8 \\
(7n - 24)/32 & \text{if } t \equiv 2, 10 \text{ mod } 24 \\
(7n + 12)/32 & \text{if } t \equiv 3 \text{ mod } 24 \\
(7n - 16)/32 & \text{if } t \equiv 4, 20 \text{ mod } 24 \\
(7n - 12)/32 & \text{if } t \equiv 5, 13 \text{ mod } 24 \\
(7n - 8)/32 & \text{if } t \equiv 6 \text{ mod } 8 \\
(7n - 4)/32 & \text{if } t \equiv 7 \text{ mod } 8 \\
(7n - 20)/32 & \text{if } t \equiv 11, 19 \text{ mod } 24 \\
(7n + 16)/32 & \text{if } t \equiv 12 \text{ mod } 24 \\
(7n + 8)/32 & \text{if } t \equiv 18 \text{ mod } 24 \\
(7n + 20)/32 & \text{if } t \equiv 21 \text{ mod } 24 
\end{cases} \quad (2)
\]
It is shown in [4] that this upper bound is tight for \( n \equiv 8 \text{ mod } 16 \). Constructions of conflict-avoiding codes achieving the upper bound (2) for the remaining cases are given subsequently in [2, 7]. The values of \( M(n, 3) \) for all even \( n \) are thereby determined by (1) and (2).

Known results for \( M(n, 3) \) for odd \( n \) is very few so far. Momihara [8] gives a necessary and sufficient condition for the optimality of equi-difference CAC(\( n, 3 \)) for large prime length. In [5], \( M(n, 3) \) is computed for all \( n \leq 100 \) except 31, 43, 57, 73, 89, 93 and 99. A table of equi-difference CAC of weight 3 can be found online at [11]. Studies of CAC in for weight larger than three can be found [9, 10].

2. CAC and Hypergraph Matching

The letter \( n \) will denote an odd and positive integer from now on. We observe that \( \Delta_1(\mathcal{A}) \) is closed under negation for any \( \mathcal{A} \subseteq \mathbb{Z}_n \). For conciseness, we can identify \( i \) and \( n - i \) in \( \mathbb{Z}_n \), and consider only the differences in the range \( \{1, 2, \ldots, (n - 1)/2\} \). Let \( m \) denote the integer \((n - 1)/2\), and \([m]\) be the set \( \{1, 2, \ldots, m\} \). Given a codeword \( \mathcal{A} \in \mathcal{P}_3(\mathbb{Z}_n) \), define
\[
\Delta_2(\mathcal{A}) := \Delta(\mathcal{A}) \cap [m].
\]
We borrow the terminology in [2] and call \( \Delta_2(\mathcal{A}) \) the \textit{halved difference set}, and note that 0 is excluded in \( \Delta_2(\mathcal{A}) \). The elements in \( \Delta_2(\mathcal{A}) \) can be interpreted as the equivalence classes of the non-zero differences in \( \mathcal{A} \) under the relation \( \sim \) defined by \( i \sim j \text{ iff } i = j \text{ or } i \equiv -j \text{ mod } n \). Because the difference sets of two distinct codewords must be disjoint, the construction of optimal CAC amounts to packing sets of the form \( \Delta_2(\mathcal{A}) \), for \( \mathcal{A} \in \mathcal{P}_3(\mathbb{Z}_n) \), in the set \( [m] \). We re-formulate the problem into a hypergraph matching problem as follows.

In general, a \textit{hypergraph} \( H = (V, E) \), where \( V \) is the vertex set and \( E \) is a collection of subsets of \( V \), called the hyperedges. A matching of hypergraph \( H \) is a collection of mutually disjoint hyperedges, and is said to be optimal if it is a matching that contains the largest possible number of hyperedges in \( H \). Let \( \nu(H) \) be the size of an optimal matching of \( H \), called the \textit{matching number} of \( H \).

For odd \( n \), let \( H(n) \) be a hypergraph on vertex set \([m]\). The edge set is a collection of subsets of \([m]\) given by
\[
E = \{ \Delta_2(\mathcal{A}) \subset [m] : \mathcal{A} \in \mathcal{P}_3(\mathbb{Z}_n) \}.
\]
We summarize the size of a hyperedge in the following lemma.

\textbf{Lemma 1.} For odd \( n \), we have
\[
|\Delta_2(\mathcal{A})| = \begin{cases} 
1 & \text{if } \mathcal{A} = \{0, a, 2a\} \text{ is centered and } a = n/3, \\
2 & \text{if } \mathcal{A} = \{0, a, 2a\} \text{ is centered and } a \neq n/3, \\
3 & \text{if } \mathcal{A} \text{ is non-centered.}
\end{cases}
\]

The proof is straightforward and omitted.

A non-centered codeword \( \mathcal{A} \) is associated with a hyperedge \( \Delta_2(\mathcal{A}) \) of size 3. We will use \( G(n) \) to denote the subgraph of \( H(n) \) obtained by removing all hyperedges of size 3. So, \( G(n) \) can be interpreted as a graph in the ordinary sense, possibly with a self-loop on the vertex \( n/3 \) if \( n \) is a multiple of 3. The following is a mere re-statement of the main problem in terms of hypergraph.

\textbf{Theorem 2.} For odd integer \( n \), we have \( \nu(H(n)) = M(n, 3) \) and \( \nu(G(n)) = M^e(n, 3) \).

For example, consider the following seven codewords of length \( n = 31 \):
\[
\mathcal{C} = \{ \{0, 2, 5\}, \{0, 4, 8\}, \{0, 6, 12\}, \{0, 7, 14\}, \\
\{0, 9, 18\}, \{0, 10, 20\}, \{0, 15, 30\} \}.
\]
The first codeword \( \{0, 2, 5\} \) is non-centered, and
\[
\Delta_2(\{0, 2, 5\}) = \{2, 3, 5\}. 
\]
The remaining codewords are centered. The corresponding halved difference sets are
\[
\Delta_2(\{0, 4, 8\}) = \{4, 8\}, \quad \Delta_2(\{0, 6, 12\}) = \{6, 12\},
\Delta_2(\{0, 7, 14\}) = \{7, 14\}, \quad \Delta_2(\{0, 9, 18\}) = \{9, 13\},
\Delta_2(\{0, 10, 20\}) = \{10, 11\}, \quad \Delta_2(\{0, 15, 30\}) = \{15, 1\}.
\]
We see that the halved difference sets of the seven codewords are distinct. They form a CAC of weight 3 and length 31. The resulting matching of graph \( H(31) \) is shown in Fig. 1. The hyperedge corresponding to the codeword \( \{0, 2, 5\} \) is shown as an ellipse. The other six edges of size two are drawn in thick lines (in red color).

The CAC in (3) is an optimal CAC\((31, 3)\). Indeed, as the length \( n = 31 \) is not a multiple of 3, all hyperedges in \( H(31) \) have at least 2. The maximal number of disjoint hyperedges is thus \( \lfloor 15/2 \rfloor = 7 \). We thus have \( \nu(H(31)) = M(31, 3) = 7 \). We note that the underlying graph \( G(31) \) consists of three 5-cycles. If hyperedge is not allowed, or equivalently if non-centered codeword is ignored, the largest matching contains six edges only, i.e., \( \nu(G(31)) = M^c(31, 3) = 6 \).

The subgraph \( G(n) \) is used in [5] in the study of equi-difference CAC. Two vertices \( x \) and \( y \) in \( G(n) \) are connected by an edge if \( y \equiv 2x \mod n \) or \( y = -2x \mod n \). It is known that the graph \( G(n) \) is regular with degree 2, with the possible exception that when \( n \) is a multiple of 3, the vertex \( n/3 \) in \( G(n) \) is an isolated vertex. Therefore, \( G(n) \) consists of disjoint cycles. The problem of finding \( M^c(n, 3) \) amounts to obtaining a matching in a union of disjoint cycles.

The following theorem is the key in all known results about \( M(n, 3) \) for odd \( n \) in the literature so far. We write \( d/\lceil n \rfloor \) if \( d \) is a divisor of \( n \), and write \( d/n \) otherwise. The number of odd-cycles in \( G(n) \) is denoted by \( N_{odd}(n) \). (A self-loop is considered as an odd-cycle of length 1.)

**Theorem 3** ([5], [6]). Let \( n \) be an odd integer.

1. If \( 3 \nmid n \) and \( N_{odd}(n) \leq 2 \), then
\[
M(n, 3) = M^c(n, 3) = (n - 1 - 2N_{odd}(n))/4.
\]

2. If \( 3 \mid n \) and \( N_{odd}(n) \leq 2 \), then
\[
M(n, 3) = M^c(n, 3) = 1 + (n - 1 - 2N_{odd}(n))/4.
\]

In both cases in Theorem 3, the optimal number of codewords \( M(n, 3) \) can be achieved by equi-difference CAC.

**3. A Sufficient Condition for Optimal CAC**

In this section we give an extension of Theorem 3 by including non-equidifference codewords. In hypergraph \( H(n) \), where hyperedges of size 3 are included, we also partition the vertices into cycle as in the subgraph \( G(n) \). When we talk about a “cycle” in \( H(n) \), we mean a cycle formed by some edges of size 2.

We first consider an example of CAC with length \( n = 129 \). The graph \( G(129) \) is shown in Fig. 2. There are 9 cycles of length 7. Each cycle is displayed horizontally in a row. As 129 is divisible by 3, there is a single isolated vertex with label 43. In each cycle, we can pack 3 edges of size 2. So, the maximal number of equi-difference codewords, including the codeword \( \{0, 43, 86\} \), is 9 x 3 + 1 = 28, and thus \( M^c(129) = 28 \). In each cycle, there is a vertex which is not covered by the edges of size 2. We can include three more hyperedges of size 3, for example, \( \{2, 3, 5\} \), \( \{7, 11, 18\} \) and \( \{13, 38, 51\} \). We have a CAC of length 129 with 31 codewords — 28 equi-difference codewords generated by 4, 6, 10, 14, 15, 16, 21, 22, 24, 25, 26, 27, 29, 31, 33, 34, 35, 36, 40, 41, 43, 45, 46, 53, 55, 56, 60, 64, 68, and three non-equidifference codewords \( \{0, 2, 5\} \), \( \{0, 7, 18\} \) and \( \{0, 13, 51\} \). This is indeed optimal, as we shall see in the next theorem.

**Theorem 4.** Let \( n \) be an odd integer and let \( N_{odd}(n) \) be the number of odd-cycles in \( G(n) \). If we can find \( N_{odd}(n)/3 \) mutually disjoint hyperedges in \( H(n) \) of size 3 lying across 3 \( \lfloor N_{odd}(n)/3 \rfloor \) cycles of odd length \( \geq 3 \), then
\[
M(n, 3) = \begin{cases} 
\frac{n-1-2N_{odd}(n)}{4} + \left\lfloor \frac{N_{odd}(n)}{3} \right\rfloor & \text{if } 3 \nmid n, \\
\frac{n-1-2N_{odd}(n)}{4} + \left\lfloor \frac{N_{odd}(n)-1}{3} \right\rfloor & \text{if } 3 \mid n.
\end{cases}
\]

The meaning of “hyperedges \( E_1, E_2, \ldots, E_\mu \) lying across cycles of odd length” is: each element in the union \( E_1 \cup \ldots \cup E_\mu \) is in an odd-cycle, and no two distinct elements in \( E_1 \cup \ldots \cup E_\mu \) are in the same odd-cycle. An example of 3 hyperedges lying across 9 cycles is shown in Fig. 2.

**Proof.** Let \( \mathcal{E} \) be an optimal CAC of length \( n \) and weight 3. It corresponds to an optimal matching in the hypergraph \( H(n) \). Let \( x_1, x_2, x_3 \) be the number hyperedges of size 1, 2 and 3 respectively in this matching. The total number of codewords in \( \mathcal{E} \) is \( x_1 + x_2 + x_3 \).

The value of \( x_1 \) is equal to 0 when \( n \) is not a multiple of 3. However, if \( n \) is a multiple of 3, say \( n = 3j \) for some
integer \( j \), then we can find an optimal matching of \( H(n) \) containing the hyperedge \( \{ j \} \) of size 1. Let \( \mathcal{M} \) be a maximal matching. If the vertex \( j \) is not covered by any hyperedge in \( \mathcal{M} \), then we can add the hyperedge \( \{ j \} \) to \( \mathcal{M} \) and increase the number of hyperedges in the matching by one, contradicting the assumption that \( \mathcal{M} \) is optimal. If the vertex \( j \) is covered by some other hyperedge in \( \mathcal{M} \) already, we can remove this hyperedge from \( \mathcal{M} \) and replace it by \( \{ j \} \). The resulting collection of hyperedges is a matching with the same number of hyperedges as in \( \mathcal{M} \), and is hence optimal. We can hence assume without loss of generality that \( x_1 = 1 \) if \( 3 | n \), and \( x_1 = 0 \) otherwise.

Because the hyperedges in a matching must be disjoint, we have \( x_1 + 2x_2 + 3x_3 \leq m \). As there are \( N_{\text{odd}}(n) \) odd-cycles, we have

\[
x_2 \leq \frac{(m - N_{\text{odd}}(n))/2}{2}. \quad (5)
\]

Suppose that \( 3 \nmid n \), i.e., \( x_1 = 0 \). By adding (5) to \( 2x_2 + 3x_3 \leq m \), we obtain

\[
3x_2 + 3x_3 \leq \frac{3m - 3N_{\text{odd}}(n)}{2} + N_{\text{odd}}(n).
\]

Using the fact that \( m - N_{\text{odd}}(n) \) is even, we divide both sides of the above inequality by 3 and get

\[
|\mathcal{E}| = x_2 + x_3 \leq \frac{m - N_{\text{odd}}(n)}{2} + \left\lfloor N_{\text{odd}}(n)/3 \right\rfloor.
\]

The case where \( n \) is divisible by 3 can be derived similarly. The details are omitted.

In view of Theorem 4, we propose the following algorithm to construct CAC of odd length and weight 3. We first construct an auxiliary hypergraph \( G' \) with \( N_{\text{odd}}(n) \) vertices. Each vertex in \( G' \) is associated with an odd-cycle in \( H(n) \). For each hyperedge in \( H(n) \) lying across three distinct odd-cycles, we put a hyperedge of size 3 covering the three corresponding vertices in the auxiliary graph \( G' \). Then we apply any available hypergraph matching algorithm on the auxiliary graph \( G' \). If we can find \( \left\lfloor N_{\text{odd}}(n)/3 \right\rfloor \) mutually disjoint hyperedges, then \( M(n, 3) \) is given as in Theorem 4. Nevertheless, if less than \( \left\lfloor N_{\text{odd}}(n)/3 \right\rfloor \) mutually disjoint hyperedges are found, then we have a CAC which may or may not be optimal.

The following theorem covers some cases where non-centered codewords are provably not useful in constructing optimal CAC. In these cases, \( M(n, 3) \) is strictly less than the expression in (4).

**Theorem 5.** For \( a \geq 1 \), we have

(i) \( M(3^a, 3) = (3^a - 2a + 3)/4 \).

(ii) \( M(7^a, 3) = (7^a - 2a - 1)/4 \).

**Proof.** (sketch) In \( H(3^a) \), there are \( a \) odd-cycles, and the cycle lengths are \( 1, 3, 9, \ldots, 3^{a-1} \). (We check that \( m = (3^{a-1} - 1)/2 = 1 + 3 + \ldots + 3^{a-1} \). The longest cycle contains all integers in \( [m] \) which are not divisible by 3. The second longest cycle contains integers in \( [m] \) which are divisible by 3, but not divisible by 9, and so on.

A characteristic of the hypergraph \( H(3^a) \) is that there is no hyperedge of size 3 which lies across three distinct cycles. We can see this by means of the 3-adic valuation function, \( v_3(x) \), defined as the largest exponent \( e \) such that \( 3^e | x \) but \( 3^{e+1} \nmid x \). The 3-adic valuation function has the property that \( v_3(x) \neq v_3(y) \Rightarrow v_3(x+y) = \min\{v_3(x), v_3(y)\} \). A vertex from a cycle of length \( 3^a - 1 \) has 3-adic valuation \( i \), for \( i = 0, 1, \ldots, a - 1 \). If we take a vertex from a 3\( i \)-cycle and another vertex from a 3\( i+1 \)-cycle, the valuation of the sum or difference of this two numbers is equal to the minimum of \( i \) and \( j \), and hence must belong to one of these two cycles. This proves that it is impossible to have a hyperedge of size 3 which is a transverse of three distinct cycles.

A hyperedge of size 3 either (i) covers three vertices in a cycle or (ii) covers two vertices in a cycle and one vertex in another cycle. If there is any hyperedge of size 3 in an optimal CAC(3, 3), we can replacing it with a centered codeword, without affecting the number of codewords. In other words, among those CAC of length \( 3^a \) with the largest number of codewords, there is one which contains equi-difference codewords only. Hence \( M(3^a, 3) = M^e(3^a, 3) \). The size of the optimal equi-difference CAC of length \( 3^a \) can be calculated by Theorem 3, and is equal to \( (3^a - 2a + 3)/4 \).

The proof for length \( 7^a \) is similar. \( \square \)

4. Numerical Results

The values of \( M(n, 3) \) for odd \( n < 500 \) are given by (4) except \( n = 81, 189, 243, 343, 405, 441 \). Using Theorem 5 and argument similar to the proof of Theorem 5, we can show that \( M(81, 3) = 19, M(189, 3) = 47, M(243, 3) = 60, M(343, 3) = 85, M(405, 3) = 101 \), and \( M(441, 3) = 110 \). Thus the values of \( M(n, 3) \) for odd \( n < 500 \) are determined.

Maximal CAC of odd length for \( n < 100 \) is given in Table 1.

**References**


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**Table 1** Maximal CAC of weight 3 and odd length \(n < 100\)